Final Exam Solutions

1. Let f be a differentiable function with $f(2,5) = 9$ and $\nabla f(2,5) = \langle 1, -3\rangle$.

   (a) Find the directional derivative of f at $(2,5)$ in the direction of the point $(1,7)$. (5 pts)

   The unit vector in the direction of $(1,7)$ is $\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$. Dot with the gradient to get the directional derivative is $\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle \cdot \langle 1, -3\rangle = -\frac{7}{\sqrt{5}}$.

   (b) Find a reasonable approximation for $f(1.9, 5.1)$. (5 pts)

   $f(1.9, 5.1) \approx f(2,5) + f_x(2,5)(1.9-2) + f_y(2,5)(5.1-5) =$
   $9 + (1)(-.1) + (-3)(.1) = 8.6$

   (c) Let $z = f(4e^t - s, \sin(st) + 5)$. Find $\partial z/\partial t$ when $s = 2, t = 0$. (10 pts)

   Let $x = 4e^t, y = \sin(st) + 5$. By the chain rule,
   $\partial z/\partial t = f_x(x,y)(4e^t) + f_y(x,y)(s\cos(st))$. At $s = 2, t = 0$ we have that $x = 2, y = 5$ so $\partial z/\partial s = f_x(2,5)(4) + f_y(2,5)(2) = 1(4) + (-3)(2) = -2$.

2. Evaluate the limit or show that it does not exist. (10 pts)

$$\lim_{(x,y)\to(0,0)} \frac{xy^4 - x^4y}{x^5 + y^5}$$

   Along the paths $x = 0, y = 0, y = x$ the limit is 0, however along $y = 2x$ the limit is $\frac{14}{33}$ so the limit does not exist.

3. Find the absolute maximum and minimum of $f(x,y,z) = 6x + 2y + z^2$ on the paraboloid $3x^2 + y^2 + 4z^2 = 100$. (15 pts)

   This is a closed and bounded region so there is a max and min. There is no interior so just check for critical points on the paraboloid using Lagrange multipliers. The Lagrange multiplier equations are $6 = \lambda 6x, 2 = \lambda 2y, 2z = \lambda 8z, 3x^2 + y^2 + 4z^2 = 100$. The first and second equations simplify to $x = 1/\lambda, y = 1/2\lambda$ so $x = y$. The third equation gives us that $z = 0$ or $\lambda = 1/4$. Consider each case and use the fourth equation to get that there are 4 critical points: $(4,4,3)$, $(4,4,-3)$, $(5,5,0)$, and $(-5,-5,0)$. Plugging these into f we get $f(4,4,3) = 41$, $f(4,4,-3) = 41$, $f(5,5,0) = 40$, $f(-5,-5,0) = -40$ so the maximum is 41 and the minimum is -40.

4. Compute $\int_C (e^{xy} + xye^{xy})dx + (x^2e^{xy})dy$ where C is the curve consisting of the two line segments from $(0,0)$ to $(3,3)$ and from $(3,3)$ to $(7,0)$. (10 pts)

   This can be done two different ways. If $F = \langle P, Q\rangle = \langle (e^{xy} + xye^{xy}), (x^2e^{xy})\rangle$ then $P_y = 2xe^{xy} + x^2ye^{xy}, Q_x = 2xe^{xy} + x^2ye^{xy}$ so F is conservative. It has potential function $f(x, y) = xe^{xy}$ so by the fundamental theorem of line integrals, the integral is $f(7, 0) - f(0, 0) = 7$.

The other way to do this is to close the region with the line segment from $(0, 0)$ to $(7, 0)$ and use Green's theorem. The integral over the whole triangle is 0 by Green's Theorem as $Q_x - P_y = 0$. The curve from $(0, 0)$ to $(7, 0)$ can be parametrized as $x = t, y = 0, 0 \le t \le 7, dx = dt, dy = 0$ so the integral over this line segment is $\int_0^7 1\ dt = 7$. Combine these two facts to get that the integral is 7.

5. Let D be a region on the xy-plane. Let S be the part of the plane $-6x + 2y + 2z = 7$ which lies above or below the region D, (i.e. points on the plane $-6x + 2y + 2z = 7$ with (x, y) in D). If the area of S is 14, find the area of D. (10 pts)

   The surface S can be parametrized as $r(x, y) = \langle x, y, (7/2) + 3x - y\rangle$ where the possible (x, y) values are exactly those in D. Then $r_x \times r_y = \langle -3, 1, 1\rangle$ so $|r_x \times r_y| = \sqrt{11}$. Using the surface area formula we get that the surface area of S is $A(S) = \iint_D \sqrt{11}\ dA = \sqrt{11}A(D)$ where $A(D)$ is the area of D. Set $A(S) = 14$ and solve for $A(D)$ to get $A(D) = 14/\sqrt{11}$.

6. Let S be the boundary of the region which is both inside the sphere $x^2 + y^2 + z^2 = 2$ and above the cone $z = \sqrt{x^2 + y^2}$. Find $\iint_S F \cdot\ d\mathbf{S}$ where $F(x, y, z) = \langle e^{\cos(yz)}, 2yz + 7x^3, 2z^2\rangle$. (15 pts)

   Use the divergence theorem. The divergence of F is $6z$. The region can be set up in either cylindrical or spherical coordinates and two set-ups are the following:

$$\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} 6zr\ dzdrd\theta$$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} 6\rho^3\sin(\phi)\cos(\phi)\ d\rho d\phi d\theta\ .$$

   The value of the integral is 3π.

7. Find $\int_C F \cdot dr$ where C is the intersection of the plane $z = 3 - 3x + 2y$ and the cylinder $x^2 + y^2 = 1$ oriented clockwise when viewed from above and $F(x, y, z) = \langle y^2 + \sin(x^2), xz, 5x\rangle$. (20 pts)

   Use Stokes Theorem with S the part of the plane which is inside the cylinder, oriented down. S can be parametrized as $r(x, y) = \langle x, y, 3 - 3x + 2y$ where $x^2 + y^2 \le 1$. Then $r_x \times r_y = \langle 3, -2, 1\rangle$ and we change this to $\langle -3, 2, -1\rangle$ to match the orientation. The curl of F is $\langle -x, -5, z - 2y\rangle$ and on S this is $\langle -x, -5, 3 - 3x\rangle$. The dot product of the curl and the normal vector is $-13 + 6x$ so the integral is $\iint_{x^2+y^2\le 1} -13 + 6x\ dA$. To evaluate, switch to polar to get $\int_0^{2\pi}\int_0^1 -13r + 6r^2\cos(\theta)\ drd\theta = -13\pi$.